

How to make all Natural Numbers using one only of digits 1, 2, 3, 4

Notice that:

$$\sqrt{2} = 2^{\frac{1}{2}}$$

$$\sqrt{\sqrt{2}} = 2^{\frac{1}{4}}$$

$$\sqrt{\sqrt{\sqrt{2}}} = 2^{\frac{1}{8}}$$

So:

$$\log_2 \sqrt{2} = \frac{1}{2} = \left(\frac{1}{2}\right)^1$$

$$\log_2 \sqrt{\sqrt{2}} = \frac{1}{4} = \left(\frac{1}{2}\right)^2$$

$$\log_2 \sqrt{\sqrt{\sqrt{2}}} = \frac{1}{8} = \left(\frac{1}{2}\right)^3$$

So:

$$\log_{\frac{1}{2}}(\log_2 \sqrt{2}) = 1$$

$$\log_{\frac{1}{2}}(\log_2 \sqrt{\sqrt{2}}) = 2$$

$$\log_{\frac{1}{2}}(\log_2 \sqrt{\sqrt{\sqrt{2}}}) = 3$$

Then:

$$\log_{\sqrt{\frac{1}{4}}}(\log_{\sqrt{3}} \sqrt{2}) = 1$$

$$\log_{\sqrt{\frac{1}{4}}}(\log_{\sqrt{3}} \sqrt{\sqrt{2}}) = 2$$

$$\log_{\sqrt{\frac{1}{4}}}(\log_{\sqrt{3}} \sqrt{\sqrt{\sqrt{2}}}) = 3$$

Etc.